Boolean Algebra and the Yijing*

Dr Andreas Schöter

1. Introduction

This paper is concerned with the logical and structural properties of the gua, interpreted as the symbolic representation of situations. I begin with some background discussion that will hopefully give the reader some indication of my own particular approach to this great work.

1.1 The Nature of this Essay

I am aware that my preoccupation with the structural aspects leaves one half of the Yi untouched. The Yi Jing is the probably the richest, most articulated combination of the intuitive and the rational aspects of the human heart/mind ever to have been formulated. However, this essay focuses almost completely on the rational, structural aspects of the work.

This is a deliberate choice on my part and stems from two reasons.

Firstly, I believe that this aspect of the work has received little serious attention in the west, especially in the context of contemporary information theory. Most of the written material available is either historical, tracing the conceptual and textual development of the book; or presents translations and interpretations of the classical texts associated with the gua.

Secondly, pragmatically, this is the aspect of the Yi that I am best suited to address.

1.2 The Yi Jing in the Information Age

There would seem to be a significant degree of interest in the Yi Jing from computer literate people. The number of web sites devoted to the changes is substantial (see the web site listed as [ICh98] in Section 7 for a comprehensive selection of cross-references) and there are a wide range of Yi Jing consultation programs available, as freeware, shareware and commercial software. There is also a lively e-mail discussion list called Hexagram-8 devoted to the topic with an extensive archive of past discussions [Hex98].

We could ask why this is the case. Kirk McElhearn has referred to the Yi Jing as the first example of hypertext. There is a lot of truth to this analogy: if each hexagram is a node, then the changing lines are links between those nodes. Thus, we might almost

---

*This paper was first published in *THE ORACLE: THE JOURNAL OF YIJING STUDIES*, Vol 2, No 7, Summer 1998, pp19–34. ISSN 1463-6220.

expect its natural home to be the world wide web. An excellent example of a web-based Yi Jing is Chuck Polisher's *I Ching Lexicon* which uses the facilities of HTML to link a verbatim translation of the book's text to the definitions of the individual characters [Pol98]. However, if we look closely, the Yi's embodiment of hypertext linking is richer than that currently encodable using standard HTML, and no sites that I am aware of yet attempt to capture this aspect of the changes in their on-line representations.

In addition to its hypertext nature, the Yi is a system of symbols based on a binary representation. Leibniz was the first to realize this connection in the west when the Jesuit missionary Bouvet (himself a mathematician) sent him a copy of Shao Yong's "Prior Heaven" arrangement of the hexagrams in 1702 (see [deF97, pp156-158]). There are clear resonances between the symbols of the Yi and Leibniz's binary calculus, and thus with today's digital technology. It is this obvious connection that lead me to explore the application of Boolean algebra and computation tools to the structures of the gua.

1.3 Situations

We are interested in *situations*; in their internal dynamics and in the dynamics of change from one situation to another. Or rather, given that the variety of concrete situations is endless, we are interested in *types* of situations; the significant features that remain, in some way, invariant across that variety. We are, therefore, interested in the abstract features of our manifest reality.[1]

There has been a growing trend in modern formal logic to look beyond the simplistic mechanisms of the last century. Barwise and Perry make the following comment [B&P83, p7]:

> Reality consists of situations — individuals having properties and standing in relations at various spatiotemporal locations. We are always in situations; we can see them, cause them to come about, and have attitudes towards them.

They go on to develop a formal theory of situations that encompasses these entities and facilitates reasoning about them and the connections between them.

I contend that the Yi Jing provides us with a system of notation for situations. It provides us with a finite set of primitive elements (the bigrams, trigrams and hexagrams) whose properties, through study and reflection, we may hope to grasp. These elements can be combined, and the resulting interaction of their properties provide the dynamic description we seek. This essay provides an algebraic basis for investigating the Yi and, in this sense, is more in the tradition of the *Image and Number* school of interpretation than the *Meaning and Principle* school. The latter school saw the actual texts of the Yi Jing as the primary path to understanding, whilst for Shao Yong, a prominent exponent of the former school, "mathematics were the purest form of representing the process by which the phenomenal universe emerged from a unitary state of Being." [deF97 p129].

The Notational Engineering Laboratory [NEL98] say that "notational systems do not merely represent abstractions, they discover and then tokenize them." That is, the language of abstraction that we use to describe our reality can give us new insights into that reality. Mathematics is the primary example of this in the west. It has proven itself as the cornerstone of the physical sciences. It is my belief that the Yi is about exactly this. The opening chapters of the *Dazhuan* speak to this theme directly: "the holy sages instituted the hexagrams, so that phenomena might be perceived therein." (*Ta Chuan*, Chapter II, verse 1; [Wil83 p287])

In support of this approach to the Yi, de Fancourt says "the cosmology described in the Great Treatise is based on the trigrams and hexagrams, not the text of the Changes. The ancient sages created these 'images' so that the mysterious processes of the universe could be fathomed; the text is merely appended to elucidate their meaning." [deF97, pp65-66]

On a similar theme, McElhearn's excellent paper "The Key to the Yi Jing" talks about the role of schemata in understanding the Yi, although he is mostly concerned with developing a deep understanding of the existing, original texts by reconstructing the linguistic and cultural schemata of those who created them. He says "each situation in our lives corresponds to one or more schemata, and each of the hexagrams corresponds to schemata also." [McE98, p9] The notation-based, algebraic system presented here is really only the logical consequence of that view point. Hexagrams *are* schemata, and those schemata should be amenable to direct comprehension. It is in the tradition of a direct structural representation of the situations. Further, it fits into an emerging contemporary trend of applying modern analytical techniques to this ancient system (see especially the work of Higgins [Hig98], and also Goldenberg [Gol75]).

Part of the purpose of this paper is to attempt to provide the beginnings of a notational understanding of the structures of the hexagrams that will facilitate the dynamic construction of schemata. The long term goal of the project is that there should be some kind of algebra that provides a descriptive mechanism for understanding the structure of the hexagram directly. That understanding, coupled with the imagination, can then provide the information to generate a relevant schema and apply it to the situation at hand.

1.4 Names and Notation

Before starting in on the material, I must say a word about the names and notation I have used for the gua.

Firstly, the names of the gua. Where I am quoting from a particular source, then I will use the name that the author being quoted has used in that instance. However, when I am writing from my own perspective I will, for the convenience of the reader, use the translations by Wilhelm [Wil83].

Secondly then, the notation for the gua. Because of the need for this essay to exist through time in a number of different formats, I have had to adopt a notation that enables me to write the gua without any recourse to particular graphical resources. This representation is linear and textual. Yin is written as **`0`** and yang as **`1`**. Further, whilst the gua are read from the bottom upwards, this notation is read from left to right to fit with the flow of the text. Thus, the trigram Arousing will be written **`100`**.

In addition, when I wish to indicate the extra information concerning changing lines, I shall use the usual numerical designations. Thus, **`698`** would represent an initial trigram of **`010`**, the Abyss, changing into **`100`**, the Arousing.

2. Boolean Algebra

In this section I present the details of an algebra that provides operations on the gua. Specifically I consider *combinatory operations* and *orderings*. Combinatory operations are concerned with how situations, represented as gua, can be combined together; and orderings define the sequences into which the gua can be arranged.

2.1 Combinatory Operations

In this section I introduce some standard tools from algebra and show how they can be applied to the gua. The combinatory operations of an algebra are those operations that take one or more elements and act on them to produce a new element. In particular I shall be concerned with using the kinds of *bit-wise* operations that are typical in computer programming languages. We can see that the gua are, at one level, patterns of bits; the lines are either yin or yang, **`0`** or **`1`**. Computer programming languages, at their lowest level of abstraction, are tools for manipulating patterns of bits. Table 1 shows the logical operators that we shall consider in this essay.[2]

Logical name	Interpretation	Notation
not	complement	~x
or	union	x v y
and	intersection	x & y
xor	difference	x x y

Table 1: The Combinatory Operations

The first of these is a *unary* operation in that it applies to a single gua to generate another, whilst the remaining three are *binary* operations in that they form a third gua by combining two others.

2.1.1 The Not Operator

The *not* operation changes the polarity of the lines of a gua: yin becomes yang and yang becomes yin. That is, for individual lines, **~0 = 1** and **~1 = 0**. This is often represented in a tabular form:

x	~x
0	**1**
1	**0**

Table 2: The *not* operation

The effect on structures is the same, taken line by line. For example, consider the Water: it is transformed into Fire: **~010** = **101**. Similarly the Arousing is transformed into the Gentle: **~100** = **011**. This transformation is part of the basis of the traditional circular Primal Arrangement pairing of the trigrams (*i.e.*, see [Wil83, p266]).

2.1.2 The Or Operator

For the *or* operation, if *either* (or both) of the corresponding lines in the initial gua is yang, then the result is yang. Put another way **0** ∨ **0** = **0**, and for all other pairs, the result is **1**. This is summarized in the following table:

x	y	x ∨ y
0	**0**	**0**
0	**1**	**1**
1	**0**	**1**
1	**1**	**1**

Table 3: The *or* operation

Again, the extension to structures is done line by line. For example, in trigrams, the result of combining the Arousing with Stillness gives Fire: **100** ∨ **001** = **101**.

The *or* operation can be viewed as a *union* operation. Where x ∨ y = z, then z is the result of taking the union of the yang energies in x with the yang energies in y.

2.1.3 The And Operator

The result of the *and* operation on a pair of lines is such that if *both* of the lines are yang then the result is yang, otherwise the result is yin. That is, **1** & **1** = **1**, and for all other pairs, the result is **0**. Again, it is best to represent this in a tabular form:

x	y	x & y
0	**0**	**0**
0	**1**	**0**
1	**0**	**0**
1	**1**	**1**

Table 4: The *and* operation

As with the other operations, this is ready extended to structures by taking them line by line. For example, combining the energies of the Joyous with the energies of Fire through the *and* operation gives the Arousing: **110** & **101** = **100**.

This can be seen as a *intersection* operation: that is, where x & y = z, then z is the result of taking the intersection of the yang energy in x with the yang energy in y.

It should be noted that the *and* operation is not independent. It can be defined in terms of *not* and *or* as follows:

Definition 1: And

x & y = ~(~x v ~y)

2.1.4 The Exclusive Or Operator

For the *xor* ("exclusive or") operation, the result is only **1** (yang) if exactly one of its inputs are **1**, and the result is **0** otherwise. So, **1** x **0** = **1** and **0** x **1** = **1**, but other combinations result in **0**. As before, this is best represented in a tabular format:

x	y	x x y
0	**0**	**0**
0	**1**	**1**
1	**0**	**1**
1	**1**	**0**

Table 5: The *xor* operation

This is perhaps the most interesting of the structural operations from the point of view of this essay. It can be seen as a *difference* operation. That is, where x x y = z, z is the difference between x and y. Considering its action on trigrams, the Abyss would represent the difference between the Arousing and the Joyous: **100** x **110** = **010**.

Again, the *xor* operation is not independent. It can be defined in terms of the preceding operators as follows:

Definition 2: Exclusive Or

x x y = (~x & y) v (x & ~y)

2.2 A Boolean Algebra for Gua

We can now define a Boolean algebra of structures as follows [Wii87, p130]:

Definition 3: Boolean Algebra

```
<S, bot, top, v, &, ~>
```

Here `S` is a set of structures. This could be the set of two lines **`{0, 1}`**, or the set of four bigrams **`{00, 10, 01, 11}`**, or the set of eight trigrams, or so on. The items `bot` and `top` are special elements from `S`, such that the following rules always hold:

Definition 4: Boolean Laws

1 Identity laws:
   1.1 `x v bot = x`
   1.2 `x & top = x`
2 Complement laws:
   2.1 `x v ~x = top`
   2.2 `x & ~x = bot`

Thus, for the trigrams, `bot = 000` and `top = 111`. And in general the `bot` element from any set `S` is that element composed only of yin lines, and the `top` element is the complement to that, the element composed only of yang lines.

At this point I should draw the reader's attention to the work of Goldenberg [Gol75]. His paper "The Algebra of the I Ching and its Philosophical Implications" has significant parallels with the work presented here.[3]

Whilst the work here originates from the perspective of computer-based algebras (see *e.g.* [Wii87]), Goldenberg presents his analyses from a more traditional mathematical algebra perspective (see *e.g.* [Whi78]). Rather than starting from the computational notion of an array of bits, and defining operations from that point, Goldenberg uses the notion of modulo arithmetic to generate his basic operators. He uses two operators in his algebra: his $x \oplus y$ is equivalent to my `x x y`, and his $x \otimes y$ is equivalent to my `x & y`.

The parallels with the system developed here are striking (see especially the detailed discussion of change in Section 3.2). However, there are also some significant differences.

The most important difference is that Goldenberg does not formally define a complement operation. This means that his algebra, as it is presented in the paper, is not capable of defining the full range of possible relationships that may obtain between the gua. Conversely, the Boolean algebra presented here is functionally complete: it can define any bit-wise relationship between the gua.

A further difference is in the choice of his operators. Whilst the need for x (Goldenberg's ⊕) is clear, when combined with only & the result is not a *Boolean* algebra. Specifically the law of distributivity fails. Without a full Boolean algebra it is not possible to define the lattice structures introduced in the next Section, and we thus lose a potentially significant tool for interpretation.

2.3 The Induced Ordering

Readers familiar with algebraic systems will realize that the combinatory operations described above induce a *partial order* on the gua. This will be written $x \leq y$ to mean x is less than or equal to y in the ordering. Formally, the ordering is defined as follows [Wii87, p135]:

Definition 5: Partial Ordering

$x \leq y$ if and only if $x \vee y = y$

The following two figures present the resulting lattices graphically for bigrams and trigrams. The basic lattice for bigrams is given in Figure 1. The important point about such structures is that the elements at higher levels are "greater" than the elements that they are connected to at the lower levels. That is, if y is above x in the diagram, and a line can be traced from x up to y, then $x \leq y$. Thus, in the bigram lattice, we can see that **11** is greater than both **10** and **01**. So we could write **10** ≤ **11** and **01** ≤ **11**. In fact **11** = **10** ∨ **01**, and the *or* operation defines upward connectedness for the lattice (as we would expect from Definition 5). Similarly, both **10** and **01** are greater than **00**. When we take **10** & **01** we get **00** — thus *and* defines downward connectedness for the lattice (as we would expect from Definition 1).[4]

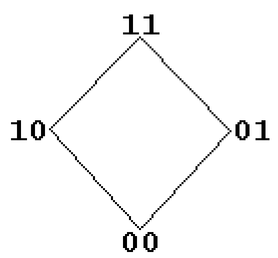

Figure 1: The Bigram Lattice

The lattice for the trigrams is given in Figure 2. This diagram can give us some interesting insights for interpreting the trigrams. We see that **000** is the bottom of the lattice, this is the trigram with the least yang and the most yin energy. As we climb up the links we encounter trigrams with increasing amounts of yang. Thus, one link up from the Receptive we find **100**, **010** and **001**. These trigrams are the minimal increase from the Receptive. At the next row we have the trigrams with two yang lines, and then at the top we have the maximally yang trigram, the Creative.

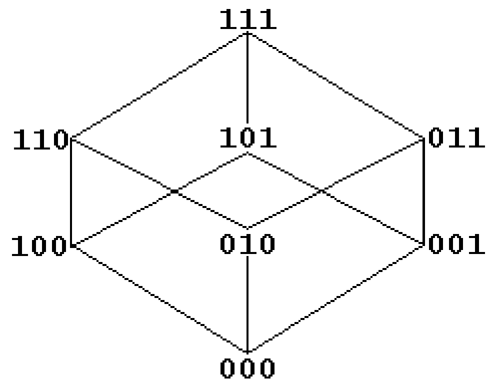

Figure 2: The Trigram Lattice

Things become interesting when we consider what this lattice structure might mean. It provides us with an interdependent way of characterizing the properties of individual trigrams in terms of the combined properties of a pair of trigrams. The following list makes the import of the lattice structure obvious:

1. The Creative is the union of complements.
2. The Joyous is the union of the Arousing and the Abyss.
3. Fire is the union of the Arousing and Stillness.
4. The Gentle is the union of the Abyss and Stillness.
5. The Arousing is the intersection of the Joyous and Fire.
6. The Abyss is the intersection of the Joyous and the Gentle.
7. Stillness is the intersection of Fire and the Gentle.
8. The Receptive is the intersection of complements.

Note that the characterizations for the Creative and the Receptive can be made more general than the other characterizations. These definitions essentially restate Clause 2 of Definition 4. We might interpret them by saying that in the Creative, as all encompassing heaven, opposites are reconciled. Whilst in the Receptive, as the material realm of earth, opposites are distinguished.

The other immediately obvious interpretation here concerns Fire **101**. This is taken to be the union of Arousing and Stillness. The Arousing is young energy erupting upwards. In contrast, Stillness is more mature and settled. By taking the union (the *or* operation) of the energy of Stillness with the energy of the Arousing we get **101**, Fire, a contained and more sustaining energy.

The other possible combinations of interpretation, moving in both directions through the lattice need to be explored.[5]

2.4 Sequences

Whilst Goldenberg uses modulo arithmetic on a line-by-line basis to define the basic operations of his algebra, I believe the correct application of modulo arithmetic in the Yi is to a gua as a whole to generate sequences.

By *sequence* I simply mean some systematic method of providing a *total order* on the gua. The minimum requirement, in this context, is that, given one gua, there is a definite method for computing which will be the next one and which is the previous one. In this context, by saying “computation”, I intend to rule out simply looking the result up in a table.

If we consider the trigrams, there are initially 8! or 40,320 possible ways of arranging the gua into sequences. Even once we eliminate repetition through rotational symmetry we are still left with 5,040 possible sequences. Cleary this is a large space to investigate. To reduce our investigation to more manageable proportions I begin by focusing on simple counting methods for generating sequences.

If we treat the lines in the gua as binary digits, and the gua itself as a binary number, then there are three variables:

Definition 6: Sequence Parameters

1. whether the lower or upper line is the least significant bit;
2. whether the operation is addition or subtraction
3. the amount of the increment or decrement.

To reduce the possibilities to the minimum for the moment we will fix the values of the second and third parameters: we will only consider incrementing sequences with an amount of increment of 1. This leaves us with two possible sequences, depending on whether the top line or the bottom line is taken as the starting point.

If we take the lowest line as the least significant bit then the sequence will be:

000, **100**, **010**, **110**, **001**, **101**, **011**, **111**

For obvious reasons, I shall refer to this as the *Rising Yang* sequence. If, instead, we take the top line as the least significant bit then the sequence will be:

000, **001**, **010**, **011**, **100**, **101**, **110**, **111**

I shall refer to this as the *Sinking Yang* sequence.

Astute readers will realize that this second arrangement was known to the Chinese scholar Shao Yong; de Fancourt reproduces a non-circular *Xiantian* or “Prior

Heaven" diagram where we see exactly this arrangement of the trigrams [deF97, p128]. However, Shao Yong almost certainly did not arrive at the arrangement by binary counting, but rather by a systematic and incremental construction. Starting with a single yin line and a single yang line, we can add to each of those lines to generate the four bigrams, and then add again to generate the trigrams.[6] The result is the same. Shao Yong arrived there through what amounts to a tree construction (see Figure 3 for the tree construction). Here we see an arithmetic route to the sequence.

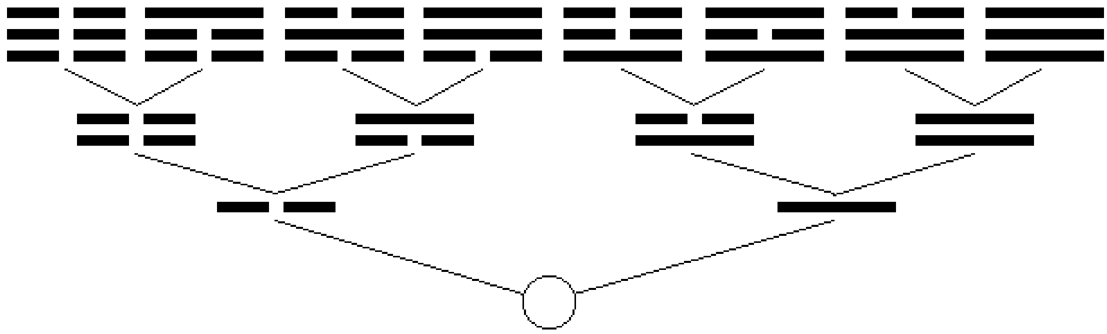

Figure 3: Shao Yong's Xiantian Diagram

3. Some Applications

It is informative to consider how the tools introduced above can be used to provide a systematic analysis of some ideas traditionally expressed in other ways.

We have already seen briefly that the *not* operation can be used to describe the relationship between the pairs of trigrams in the Primal arrangement, and that the Sinking Yang sequence matches Shao Yong's *Xiantian* development. Now I shall apply these techniques to look first at two relationships used by Cleary, and then at a method of representing change.

3.1 Cleary's Relations

Cleary introduces two sets of correspondences in the appendix to his *Pocket I Ching* [Cle92]: the first he calls the *Primal Correlate* and the second the *Structural Complement*.

3.1.1 The Structural Complement

The most straightforward of these relationships to analyze is the Structural Complement. This is simply a direct application of the *not* operation. So, the following simple definition will suffice.

Definition 7: Structural Complement

```
C  = ~X
```

Thus, the Structural Complement of Revolution (hexagram 49) is Youthful Folly (hexagram 4). If we look at this example in terms of the gua, we see that Revolution is **101110**. The complement of this is ~**101110** = **010001**, and this is hexagram 4, Youthful Folly.

3.1.2 The Primal Correlate

The Primal Correlate relationship is more complex to analyze. This is based on Shao Yong's "Old Family Mandala" arrangement of the hexagrams, (see, for example, *The Eight Lectures on the I Ching* [Wil82, Figure 2]). This circular arrangement can be seen as an extension of the Prior Heaven arrangement of the trigrams to the hexagrams. Thus, each hexagram is paired with its Structural Complement at the diametrically opposite point on the circle. Further, the Receptive is placed at the bottom of the circle (therefore putting the Creative at the top). Then, moving anticlockwise from the Receptive, the sequence is that of the first half of Sinking Yang:

000000, **000001**, **000010**, **000011**, **000100** ... **011111**

which takes us around to the Creative at the top. From that point, continuing anticlockwise, we have:

111111, **111110**, **111101**, **111100**, **111011** ... **100000**

which brings us back to the Receptive at the bottom.

Cleary's Primal Correlate relation can then be constructed by taking these two half sequences, reversing one of them, and placing them side by side:[7]

```
000000, 000001, 000010, 000011 ... 011101, 011110, 011111
        100000, 100001, 100010 ... 111100, 111101, 111110, 111111
```

Thus, the Primal Correlate of Contemplation **000011** (hexagram 20) is Difficulty at the Beginning **100010** (hexagram 3). The question that we now have to deal with is how to take this essential geometric relationship between sequence sections, and render it in a algebraic, computational form. Without presenting all of the details of the working I shall show how this relationship can be expressed.

First I shall present some auxiliary notation. Let Σ stand for the Sinking Yang sequence, and let Σ_N refer to the hexagram at position N in that sequence. Let X be the hexagram we are starting with, and let P be the Primal Correlate hexagram we are trying to compute.

Definition 8: The Primal Correlate

Where X is Σ_N:

1. if X = **111111** then P = **000000**
2. if X = **000000** then P = **111111**

3. if X & **100000** = **100000** then P is Σ_{N-31}
4. if X & **100000** = **000000** then P is Σ_{N+31}

The first two clauses of the definition deal with the two special cases of the Creative and the Receptive: it can be seen from the sequence pairing shown above that these two hexagrams protrude from either end. These two clauses capture that and pair them with each other. The third clause captures the case of mapping a hexagram in the half anticlockwise from the Creative to its Primal Correlate; the expression X & **100000** is only equal to **100000** when X is in the half of the circle anticlockwise from the Creative, then Σ_{N-31} identifies the correct hexagram in the clockwise half of circle. Similarly, the fourth clause maps a hexagram in the half clockwise from the Creative to its Primal Correlate.

As an example, consider hexagram 48 (in the King Wen sequence), the Well, **011010**. Firstly, we can determine that in the Sinking Yang sequence the Well is Σ_{26} as follows:

1. Starting with the binary representation: **011010**
2. Expanding out the elements: 32×**0** + 16×**1** + 8×**1** + 4×**0** + 2×**1** + 1×**0**
3. Then simplifying: 16+8+2
4. Gives: 26.

Next, because **011010** & **100000** = **000000**, we can see that Clause 4 of Definition 8 is the applicable rule. So, applying that clause, given that 26+31=57, we would expect the Primal Correlate of the Well to be Σ_{57}. We can determine which is gua Σ_{57} in the Sinking Yang sequence is as follows:

1. We can break down 57 as: 32+16+8+1.
2. This can be expanded to: 32×**1** + 16×**1** + 8×**1** + 4×**0** + 2×**0** + 1×**1**
3. Which gives: **111001**.

Hexagram **111001** is the Taming Power of the Great (hexagram 26 in the King Wen sequence) and if we check with Cleary's chart we see that this is, indeed, the case.

This analysis of the Primal Correlate requires a fairly complex set of clauses to capture the correct relationship. This is not surprising — the Primal Correlate is essentially a geometric relationship expressed relative to a manipulation of a basic sequence. However, the relationship *is* a systematic one, and the power of the tools presented here is demonstrated by the ability to describe Cleary's relationship.

3.2 Interpreting Change

This section applies the tools developed above to investigate a method of representing change from an algebraic perspective.

3.2.1 The Traditional Analysis

In addition to the Judgment and the Image, the classic texts of the Yi Jing associate a verse with each line, and that verse is taken to be relevant if the line is changing. If more than one line is changing, then the nature of the change is the combination of the lines' texts, possibly with the Judgment text of the other hexagram.

Firstly, consider the hexagram **678788** where the first line is changing (Deliverance, with a six in the first place, resulting in a change to the Marrying Maiden); in this simple case, the text for the first line represents the change from **010100** to **110100**. Now consider the hexagram **878988** where the fourth line is changing (Deliverance, with a nine in the fourth place, resulting in a change to the Army); similarly, the text for the fourth line represents the change from **010100** to **010000**. What about a situation where both of the previous changes occur simultaneously? That would be the hexagram represented by: **678988** (Deliverance with a six in the first place and a nine in the fourth, resulting in a change to Approach), this is the change from **010100** to **110000**, which must be represented by the combination of the texts for the first and fourth lines.

The actual rules for how to determine which texts are applicable, and which should be given most weight, gets slightly involved as the number of changing lines increases. A complete set of rules is given by de Fancourt [deF96].

3.2.2 The General Problem

From a formal perspective the problem is as follows: we have two situations that stand in a well-defined relationship to each other and we wish to be able to represent that relationship in a systematic manner. From one perspective, where G is the initial gua, and R is the gua resulting from the (possible zero) changing lines, we have a transformation $G \rightarrow R$, and we wish to know something about the nature of that transformation. This is what the line texts traditionally tell us: they explicate something about the nature of the transformation from G to R, that is, something about the nature of the arrow $\rightarrow$ that connects them.

Let us describe this transformation of one gua to another by saying that G and R are related by the process of change. If we use the symbol χ to symbolize the situation where G and R are related through that process, then I shall write $\chi(G,R)$ to formally represent the situation where G changes into R.

3.2.3 An Algebraic Analysis

Now I have to provide some content for this notation. What is the meaning of the situation where G changes into R? Firstly, my choice of the term “situation” here is deliberate. We have already suggested that a hexagram represents a situation, and now we are considering the change of one situation into another: this too is a situation, a larger situation encompassing both the original and the resulting situation. So, if the change from one situation to another is itself a situation, then we would expect the representation of the change from one hexagram into another to itself be represented by a hexagram.

If the reader thinks back to the discussion of the *xor* operation in Section 2.1.4 above, I suggested that this operation represented the difference between the two gua that it took as its parameters. Now, I want to suggest that this operator is the natural candidate for the representation of change.

Definition 9: Change

$$\chi(G,R) = G \times R$$

This definition tells us how to compute the *context* of the change from G to R.

Let us look at the implications of this definition. Consider again the first change discussed above: from **010100** to **110100**. Following Definition 9 this should be represented as follows:

$$\begin{aligned} & \chi(\mathbf{010100},\mathbf{110100}) \\ = \; & \mathbf{010100} \times \mathbf{110100} \\ = \; & \mathbf{100000} \end{aligned}$$

So, this context hexagram **100000** shows us that the first line in the initial hexagram changes to yield the resultant. This is a yin line, as indicated by the **6** in the initial formulation.

So far then, the definition seems apt. But if we look further we notice an additional property. If we start with the initial hexagram, and combine it with that context hexagram through the *xor* operation, we get the resultant:

$$\mathbf{010100} \times \mathbf{100000} = \mathbf{110100}.$$

Thus, Definition 9 shows how to compute the context of a change between an initial hexagram and its resultant, and when we apply that context to the initial hexagram, we do indeed get the resultant. These are exactly the properties we require from an algebraic analysis of change.

Let us continue with the example: consider the second change again, from **010100** to **010000**. As before, we apply Definition 9.

$$\begin{aligned} & \chi(\mathbf{010100},\mathbf{010000}) \\ = \; & \mathbf{010100} \times \mathbf{010000} \\ = \; & \mathbf{000100} \end{aligned}$$

So, under the proposed mechanism, this change would be represented by the hexagram **000100**. This shows us that the fourth line must change to yield the resultant, and this corresponds to the **9** in the original formulation. Thus, the same mechanism works to encode both yin and yang changes.

Let us now look at the combined change, from **010100** to **110000**.

$$\begin{aligned} & \chi(\mathbf{010100},\mathbf{110000}) \\ = \ & \mathbf{010100} \times \mathbf{110000} \\ = \ & \mathbf{100100} \end{aligned}$$

So, this combined change would be represented by the hexagram **100100**. Just to double check we can apply this change context to the initial hexagram:

$$\mathbf{010100} \times \mathbf{100100} = \mathbf{110000}$$

And we see that the resultant is as expected.

3.2.4 Historical Considerations

The basis of this technique is to identify the lines in the initial gua that are changing and to generate a contextual gua that encodes those lines. This kind of mechanism is not without precedence. William de Fancourt [deF97, p41] cites a text from 513BCE where a scribe called Zaimo refers to the first line of the hexagram the Creative as its Coming to Meet line, its second line as its Fellowship with Men line and so on. The hexagram Coming to Meet is **011111**, the hexagram Fellowship with Men is **101111**, so in this instance a single yin line picks out the salient line of the Creative. If we apply Definition 9 to his analysis of the first line we get the following results:

$$\begin{aligned} & \chi(\mathbf{111111},\ \mathbf{011111}) \\ = \ & \mathbf{111111} \times \mathbf{011111} \\ = \ & \mathbf{100000} \end{aligned}$$

And for the second line we see:

$$\begin{aligned} & \chi(\mathbf{111111},\ \mathbf{101111}) \\ = \ & \mathbf{111111} \times \mathbf{101111} \\ = \ & \mathbf{010000} \end{aligned}$$

Clearly Zaimo is using the same basic idea as that proposed in Definition 9 to single out the lines of a hexagram only with the polarity of the marker reversed.

This is also the place at which I should point out the direct parallel with Goldenberg's work. In his Theorem 7 he says "For any hexagram-pair there exists a third, unique, mediating hexagram which transforms either member of the pair into the other under addition" [Gol75, p163]. Now, remember that his addition operator $\oplus$ is the same as my *xor* operator x and we can see that his Theorem 7 is essentially the same as my Definition 9.

So, we have two independent points of confirmation for this technique of representing the change context. The first confirmation is indirect, and part of the early history of the Yi Jing where the same basic idea is used to identify individual lines in a hexagram. The second confirmation is direct, and part of the emerging tradition of contemporary formal analyses of the Yi. Goldenberg comes from the slightly different direction of a more purely mathematical approach, but arrives at the same mechanism for encoding change as I do, coming from a computational perspective.

4. Conclusions

This essay has explored a formal, computational analysis of the structures of the gua. More properly I should say that it has *begun* that exploration, for there is much work yet to do in this area. This is not an easy subject to tackle: it requires a detailed knowledge of some mathematical and logical concepts to grasp the exploration that is being started here. However, I believe that the fit between these areas of mathematics and the Yi is clear and not co-incidental.

It is not co-incidental because the Yi embodies structure: if one believes that the Universe is a cosmos, and that the Yi describes that Universe, then how could the Yi not encode structure? For the Yi, the starting point of that structure is the complementary relationship between yin and yang. This binary characterization is the most fundamental form of information — it is the minimal distinction, but in being minimal, it is also the easiest distinction on which computation can be built.

I do not mean to suggest that the ancient sages who constructed the Yi conceived of, or understood, the Yi as the basis of modern digital techniques. What I do suggest is that the reason those sages developed an essentially binary symbolism of situations is the same reason that the founding fathers of digital computers used a binary representation: because it is the easiest. And after all, one translation of the term “yi” is as “easy”!

5. Technical Glossary

In this section I shall offer some concise definitions of some of the technical terms used in the body of the essay. Readers who require a more involved explanation are referred to Wiitala [Wii87] for general material on Boolean algebra and partial orders, and to Davey and Priestly [D&P90] for more detailed material on lattices.

Bit — *binary digit*. The smallest unit capable of conveying information. A bit always and only takes the value of either 1, or 0. In this context we take 1 as being yang and 0 as being yin. Thus, for example, a trigram is composed of three bits.

Boolean algebra. A mathematical structure that provides a formal interpretation of the logical operator denotes by the terms *not*, *or* and *and*. Set theory is a typical example of a Boolean algebra, as are the bit-wise operators from computer programming used in this essay.

HTML — *Hypertext Markup Language*. The particular language for representing **hypertext** relationships in documents that has become dominant on the Internet.

Hypertext. The technique of linking documents together through active cross-references such that following a cross-reference takes you directly to the document where the reference is defined.

Lattice. A structure defined by a **partial order** where the points in the structure are connected by lines, and being connected by a line means that the higher element is "greater" than the lower element. The lattice will often have a top element which is greater than any other element in the set, and a corresponding bottom element which is less than any other element in the set.

Modulo arithmetic. Essentially "clock" arithmetic. Consider the face of a clock: if you add 3 to 10 you get back to 1, if you subtract 5 from 2 you get to 9. This is arithmetic modulo 12, but in principle any modulo factor could be used.

Partial order. A way of providing an order for a set of elements such that some elements may not be either greater or less than each other, hence the ordering is not **total**. Consider the trigrams ordered by degree of yangness. Clearly **`101`** has more yang than **`100`**, but both **`101`** and **`110`** have the same amount of yang so neither is greater or less than the other.

Schemata. A term used in certain cognitive theories about the mental representation of events and situations. They are data structures for representing both generic and specific concepts about the outside world.

Total order. A way of providing an order for a set of elements such that every element is either greater than or less than any other element in the set. The natural numbers ...-3,-2,-1,0,1,2,3... are the most straightforward example of a totally ordered set.

6. Notes

[1] Kirk McElhearn [McE98, p3], is the first person to use the term *situation* to refer to the underlying meaning of a hexagram in print. This is a fortunate use of terminology.

[2] The syntax used for these operators is the standard one employed in such programming languages as C and Java, rather than any of the equivalent notations used in mathematical logic. The author has developed an experimental set of programs written in the programming languages Java and Prolog that implement the operations discussed in this essay.

[3] I should note that the bulk of the work on this paper was done long before Goldenberg's work was brought to my attention by Steve Moore.

[4] A good introduction to the mathematics of such structures can be found in [D&P90]

[5] There is an interesting connection between the lattice structure and the polynomial space that Higgins investigates for the trigrams [Hig98]. Without going into details, Higgins places the trigrams in the space represented by $(a+i)^3$ which expands to:

$$a^3 + 3a^2i + 3ai^2 + i^3$$

If we compare this with the lattice in Figure 2 we see that each term in the expanded polynomial relates to a distinct layer in the lattice. This connection needs to be investigated further.

[6] Wilhelm shows how this process can be extended all the way to the generation of the hexagrams [Wil82, Figure 1].

[7] I am grateful to Frank Coolen from the Hexagram 8 discussion list for clarifying the precise geometric nature of Cleary's Primal Correlate relation; this relation originates in Cleary's book *I Ching Mandalas*, to which I do not currently have access.

7. References

Note that where the medium of publication is the world wide web, the URL from which the original document was retrieved is given. If the document contained an indication of its date of publication, then that date is used, otherwise the date of retrieval is given.

[B&P83] Jon Barwise and John Perry
Situations and Attitudes. Cambridge, Massachusetts: Bradford Books, the MIT Press, 1983. ISBN 0 262 02189 7.

[Cle92] Thomas Cleary
I Ching: the Book of Change. Shambhala Pocket Classics. Boston: Shambhala Publications, Inc., 1992. ISBN 0 87773 661 8.

[D&P90] B.A. Davey and H.A. Priestly
Introduction to Lattices and Order. Cambridge: Cambridge University Press, 1990. ISBN 0 521 36766 2.

[deF96] William de Fancourt
"Sixes and Nines — Changing Lines" in *The Oracle: the Journal of the Yi Jing Society*, Volume 1, Number 3, Spring 1996.

[deF97] William de Fancourt
Warp and Weft. In Search of the I-Ching. Freshfields: Capall Bann Publishing, 1997. ISBN 1 86163 011 9

[Gol75] Daniel S Goldenberg
"The Algebra of the I Ching and its Philosophical Implications", in *Journal of Chinese Philosophy* 2, 1975 pp149–179. D. Reidel Publishing Company.

[Hex98] Hexagram 8
http://www.apocalypse.org/pub/u/rwhe/hex8.html

[Hig98] Charlie Higgins

The Mathematics of Alchemy and the I Ching http://www.mension.com

[ICh98] I Ching Index
http://members.aol.com/IChing1/index.html

[McE98] Kirk McElhearn
"The Key to the Yi Jing" in *The Oracle: the Journal of the Yi Jing Society*, Volume 1, Number 6, Spring 1998.
Also available on the web as http://www.mcelhearn.com/key.html

[NEL98] Notational Engineering Laboratory
http://www.seas.gwu.edu/seas/institutes/nel/

[Pol98] Chuck Polisher
The I Ching Lexicon http://www.ns.net/~cpolish/DEFAULT.HTM

[Whi78] Thomas A. Whitelaw
An Introduction to Abstract Algebra. Glasgow: Blackie, 1978. ISBN 0 216 90488 9.

[Wii87] Stephen A. Wiitala
Discrete Mathematics: A Unified Approach. New York: McGraw-Hill International Editions, 1987. ISBN 0 07 100316 9.

[Wil82] Hellmut Wilhelm
Change: Eight Lectures on the I Ching. London: Routledge & Kegan Paul, 1982. ISBN 0 7100 6661 9

[Wil83] Richard Wilhelm
I Ching or Book of Changes. London: Routledge & Kegan Paul, 1983. ISBN 0 7100 9527 9